MA 335, Spring 2012

Extra Topic 1.
Help the Evil Teacher

This assignment is due *Tuesday* May 1 (update: actually, Wednesday May 2).

This is an extra assignment, worth the same as a regular homework in terms of course grade. It is not required to complete the course. If you choose to do this assignment, the grade for it will only go to the numerator of your grade.

All problems in this set are worth the same amount of points, but not necessarily have the same difficulty. There are 42 points total, 35 points is considered 100%. If you go over 35 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

No extensions will be granted under any conditions. On the other hand, if you choose to, you are welcome to submit problems in this assignment one-by-one or in any combination (for example, problems $1, 2, 6$ some time in March, then problem 5 two weeks later and then problems $3, 4$ another week later).

All material necessary to do this assignment was covered before the Spring Break.

(1) [7pt] Evil teacher wants to come up with a pair of positive integers for an exercise in Euclidean Algorithm. There are two requirements:
   (i) these numbers must be under 700 (so evil teacher can pretend to be reasonable),
   (ii) Euclidean algorithm for this pair must take the as many steps as possible (to evil teacher's twisted pleasure).

   Find such a pair of numbers and the corresponding number of steps.

   (Note: here, "step" is an equality of the form $\gcd(a, b) = \gcd(b, r)$, where $a \geq b$ and r is the remainder of a mod b. For example, $\gcd(23, 7) = \gcd(7, 2) = \gcd(2, 1) = \gcd(1, 0)$ is three steps.)

(2) [7pt] Evil teacher picked a prime number p and intends to give students a congruence of the form $ax \equiv 1 \pmod{p}$ to solve. Teacher wants to pick the coefficient a so that
   (i) $p \nmid a$ (so congruence has a solution),
   (ii) $x = 1, 2, 3, -1, -2, -3$ are not solutions (so students can't easily guess the answer).

   Find all values of p for which this is impossible.

(3) [7pt]
   (a) Evil teacher is fond of the number 29. He (she) wants to give students a problem of the form "Find the remainder of $(n - 4)!$ mod n" (for example, "Find the remainder of 26! mod 30") so that the answer is 29. Find all possible values of n.
   (b) Same question with 31 instead of 29; same question with 27 instead of 29.

— *see next page* —

(4) [7pt] There are $N > 1$ problems in a certain section of Textbook, and the evil teacher wants to pick one of them for the homework, but prefers not to think too hard, which one to pick.

So he (she) takes a counting rhyme of 33 words and counts problems in cycles $(1 - 2 - \ldots - N - 1 - 2 - \ldots)$ using this counting rhyme. But it turns out that the problem picked by the rhyme is not to his (her) liking. So evil teacher does the same again with a rhyme of 54 words. It points to the same problem. Then evil teacher does the same again, with a rhyme of 110 words, but the result is unchanged. At this point, he (she) gives up and the unfortunate problem goes into homework. What is the number of this problem?

(5) [7pt]

(a) Evil teacher wants to come up with positive integers a_1, a_2 such that the following problem has a unique answer.

> *Let f be a multiplicative number theoretic function valued in positive integers (i.e. for any positive integer n, the value $f(n)$ is a positive integer). Given that $f(20) = a_1$ and $f(50) = a_2$, find $f(100)$.*

Prove that the only choice of a_1, a_2 that satisfies such condition is $a_1 = a_2 = 1$. (Which is boring, so evil teacher has to explore further — see (b).)

(b) Find at least one triple of positive integers $a_1 > 1, a_2 > 1, a_3 > 1$ such that the following problem has a unique answer.

> *Let f be a multiplicative number theoretic function valued in positive integers. Given that $f(20) = a_1$, $f(50) = a_2$ and $f(10) = a_3$, find $f(100)$.*

(6) [7pt] Evil teacher reads Textbook problem 6.1.12(a):

> *Find the form of all positive integers n satisfying $\tau(n) = 10$. What is the smallest positive integer for which this is true?*

and wants to vastly improve it. Specifically, he (she) wants to replace 10 with a number from 2 to 15 so that the answer to the problem is maximal possible among all fourteen choices. Which number should he (she) pick and what is the answer in such case?